EA616A — Análise Linear de Sistemas
Exercícios — Transformada de Laplace

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Transformada Bilateral

- Determine $\mathscr{L}\{x(t)\}$ e o domínio de existência Ω_x para:

1. $x(t) = t^2 \exp(-4t)u(-t)$
2. $x(t) = \cos(5t)u(-t)$
3. $x(t) = t\,\mathrm{sen}(3t)u(-t)$
4. $x(t) = -\dfrac{t^m}{m!}\exp(-at)u(-t)$ (para casa)

$x(t) = t^2 \exp(-4t) u(-t)$

Propriedade: $\mathscr{L}\{x(-t)\} = X(-s), \quad -x \in \Omega_x$

$$y(t) = x(-t) = t^2 \exp(4t) u(t) \quad \Rightarrow \quad Y(s) = \frac{2}{(s-4)^3}, \quad \mathrm{Re}(s) > 4$$

$$\Rightarrow \quad X(s) = Y(-s) = \frac{2}{(-s-4)^3} = \frac{-2}{(s+4)^3}$$

$$\Omega_x = -s \in \Omega_y : \quad \mathrm{Re}(-s) > 4 \quad \Rightarrow \quad \mathrm{Re}(s) < -4$$

$x(t) = \cos(5t)u(-t)$

$$y(t) = x(-t), \quad y(t) = \cos(-5t)u(t) = \cos(5t)u(t)$$

$$\Rightarrow \quad Y(s) = \frac{s}{s^2+25}, \quad \mathrm{Re}(s) > 0$$

$$X(s) = Y(-s) = \frac{-s}{(-s)^2+25} = -\frac{s}{s^2+25}, \quad \mathrm{Re}(s) < 0$$

$x(t) = t\,\mathrm{sen}(3t)u(-t)$

$$y(t) = x(-t) = -t\,\mathrm{sen}(-3t)u(t) = t\,\mathrm{sen}(3t)u(t)$$

Propriedade: $\mathscr{L}\{tf(t)\} = -\dfrac{d}{ds}F(s), \quad s \in \Omega_y$

$$\mathscr{L}\{\mathrm{sen}(3t)u(t)\} = \frac{3}{s^2+9}, \quad \mathrm{Re}(s) > 0$$

$$Y(s) = \mathscr{L}\{t\,\mathrm{sen}(3t)u(t)\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}, \quad \mathrm{Re}(s) > 0$$

$$X(s) = Y(-s) = \frac{-6s}{((-s)^2+9)^2} = \frac{-6s}{(s^2+9)^2}, \quad \mathrm{Re}(s) < 0$$

$x(t) = \mathscr{L}^{-1}\left\{\dfrac{-2s^2+9s-22}{(s-1)^2(s-4)}\right\}, \quad 1 < \mathrm{Re}(s) < 4$

$$X(s) = \frac{-2s^2+9s-22}{(s-1)^2(s-4)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-4}$$

$$A = \left.\frac{-2s^2+9s-22}{(s-4)}\right|_{s=1} = \frac{-15}{-3} = 5, \quad C = \left.\frac{-2s^2+9s-22}{(s-1)^2}\right|_{s=4} = \frac{-18}{9} = -2$$

$$A(s-4) + B(s-1)(s-4) + C(s-1)^2 = -2s^2+9s-22 \quad \Rightarrow \quad B = 0$$

$$x(t) = 5t\exp(t)u(t) + 2\exp(4t)u(-t)$$

$x(t) = \mathscr{L}^{-1}\left\{\dfrac{8s^2 - 43s + 60}{(s-2)^2(s-5)}\right\}, \quad 2 < \mathrm{Re}(s) < 5$

$$X(s) = \frac{8s^2 - 43s + 60}{(s-2)^2(s-5)} = \frac{A}{(s-2)^2} + \frac{B}{s-2} + \frac{C}{s-5}$$

$$A = \left.\frac{8s^2 - 43s + 60}{(s-5)}\right|_{s=2} = \frac{6}{-3} = -2, \quad C = \left.\frac{8s^2 - 43s + 60}{(s-2)^2}\right|_{s=5} = \frac{45}{9} = 5$$

$$A(s-5) + B(s-2)(s-5) + C(s-2)^2 = 8s^2 - 43s + 60 \quad \Rightarrow \quad B = 3$$

$$x(t) = (-2t+3)\exp(2t)u(t) - 5\exp(5t)u(-t)$$

Transformada Unilateral: Determine

1. A solução $y(t)$ e o valor de $y(0)$ para que não haja transitório em

$$\dot{y} + 4y = 5\cos(2t)u(t)$$

2. A solução $y(t)$ para $\ddot{y} - 4y = 0, \quad y(0) = 5, \; y(1) = 10$
3. A resposta ao degrau $y_u(t)$ e à rampa $y_r(t)$ (C.I. nulas) para

$$\ddot{y} + 5\dot{y} + 6y = \ddot{x} + 11\dot{x} + 30x \quad \Rightarrow \quad (p^2 + 5p + 6)y = (p^2 + 11p + 30)x$$

4. As constantes a e b para que a solução forçada (regime permanente) da equação $\ddot{y} + 5\dot{y} + 6y = at + b$ seja dada por $y_f(t) = 5t + 10$
5. Determine o valor inicial $y(0^+)$ e o valor final $y(+\infty)$ para

$$Y(s) = \frac{6s^3 + 24s^2 + 12s + 60}{(s^2+4)(s+3)s}, \quad \mathrm{Re}(s) > 0$$

$\dot{y} + 4y = 5\cos(2t)u(t)$, $y(0)$ sem transitório

$$sY(s) - y(0) + 4Y(s) = 5\frac{s}{s^2+4} \quad \Rightarrow \quad Y(s) = \frac{5s}{(s+4)(s^2+4)} + \frac{y(0)}{s+4}$$

$$Y(s) = \underbrace{\frac{5s}{(s+4)(s^2+4)}} + \frac{y(0)}{s+4} = \underbrace{\frac{A}{s+4} + B\frac{s}{s^2+4} + C\frac{2}{s^2+4}} + \frac{y(0)}{s+4}$$

$$A = \frac{5s}{s^2+4}\Big|_{s=-4} = \frac{-20}{20} = -1 \quad \Rightarrow \quad y(0) = -A = 1 \text{ anula o transitório}$$

$$A(s^2+4) + Bs(s+4) + 2C(s+4) = 5s \quad \Rightarrow \quad B = 1,\ C = \frac{1}{2}$$

$$y(t) = \cos(2t) + \frac{1}{2}\text{sen}(2t)$$

Solução forçada de $\dot{y} + 4y = 5\cos(2t)$

$$H(s) = \frac{1}{s+4} \quad \Rightarrow \quad y_f(t) = 5|H(j2)|\cos(2t + \angle H(j2))$$

$$5\left|\frac{1}{j2+4}\right| = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}, \quad \theta = \angle\left(\frac{1}{j2+4}\right) = -\arctan(0.5)$$

Partindo de $y(t) = \cos(2t) + \dfrac{1}{2}\mathrm{sen}(2t)$, tem-se

$$\begin{aligned}
y(t) =& \frac{1}{2}\exp(j2t) + \frac{1}{2}\exp(-j2t) + \frac{1}{2}\left(\frac{1}{2j}\exp(j2t) - \frac{1}{2j}\exp(-j2t)\right) \\
=& \frac{2j+1}{4j}\exp(j2t) + \frac{2j-1}{4j}\exp(-j2t) \\
=& \frac{\sqrt{5}}{2}\left(\frac{1}{2}\exp(j2t - \arctan(0.5)) + \frac{1}{2}\exp(-j2t + \arctan(0.5))\right) \\
=& \frac{\sqrt{5}}{2}\cos(2t - \arctan(0.5))
\end{aligned}$$

$y(t)$ para $\ddot{y} - 4y = 0, \quad y(0) = 5, \; y(1) = 10$

$$s^2 Y(s) - sy(0) - \dot{y}(0) - 4Y(s) = 0 \quad \Rightarrow \quad Y(s) = \frac{sy(0) + \dot{y}(0)}{s^2 - 4}$$

$$Y(s) = \frac{sy(0) + \dot{y}(0)}{(s+2)(s-2)} = \frac{A}{s+2} + \frac{B}{s-2}$$

$$A = \frac{-2y(0) + \dot{y}(0)}{-4}, \quad B = \frac{2y(0) + \dot{y}(0)}{4}$$

$$y(t) = \left(\frac{(10 - \dot{y}(0))}{4} \exp(-2t) + \frac{(10 + \dot{y}(0))}{4} \exp(2t) \right) u(t)$$

$$y(1) = 10 \quad \Rightarrow \quad 40 = (10 - \dot{y}(0)) \exp(-2) + (10 + \dot{y}(0)) \exp(2)$$

$$\Rightarrow \quad \dot{y}(0) = \frac{40 - 10(\exp(2) + \exp(-2))}{\exp(2) - \exp(-2)}$$

$y_u(t)$ e $y_r(t)$ (C.I. nulas), $\ddot{y}+5\dot{y}+6y=\ddot{x}+11\dot{x}+30x$

$$H(s)=\frac{s^2+11s+30}{s^2+5s+6}, \quad x(t)=tu(t), \quad X(s)=\frac{1}{s^2}$$

$$Y_r(s)=\frac{s^2+11s+30}{(s^2+5s+6)s^2}=\frac{s^2+11s+30}{(s+2)(s+3)s^2}=\frac{5}{s^2}+\frac{7/3}{s}+\frac{-2/3}{s+3}+\frac{3}{s+2}$$

$$y_r(t)=\Big(5t+\frac{7}{3}-\frac{2}{3}\exp(-3t)+3\exp(-2t)\Big)u(t)$$

$$Y_u(s)=\frac{s^2+11s+30}{(s^2+5s+6)s}=\frac{5}{s}+\frac{2}{s+3}-\frac{6}{s+2}$$

$$y_u(t)=\Big(5+2\exp(-3t)-6\exp(-2t)\Big)u(t) \quad (\text{Note que } y_u(t)=py_r(t))$$

Verifique (em casa) que $y_r(t)=\displaystyle\int_0^t y_u(\beta)d\beta$ e determine $h(t)=\mathscr{L}^{-1}\{H(s)\}$

Determine a, b para que a sol. forçada de $\ddot{y}+5\dot{y}+6y = at+b$ seja dada por $y_f(t) = 5t+10$

$$H(s) = \frac{1}{s^2+5s+6}, \quad Y(s) = \frac{1}{(s+3)(s+2)}\left(\frac{a}{s^2}+\frac{b}{s}\right)$$

$$Y(s) = \frac{aH(0)}{s^2} + \frac{a\dot{H}(0)}{s} + \frac{bH(0)}{s} + \text{transitório}$$

$$Y_f(s) = \frac{5}{s^2} + \frac{10}{s} = \frac{aH(0)}{s^2} + \frac{a\dot{H}(0)+bH(0)}{s}$$

$$H(0) = \frac{1}{6}, \quad \dot{H}(0) = \frac{d}{ds}H(s)\Big|_{s=0} = \frac{-(2s+5)}{(s^2+5s+6)^2}\Big|_{s=0} = -\frac{5}{36}$$

$$\frac{a}{6} = 5 \quad \Rightarrow \quad a = 30, \qquad \frac{30(-5)}{36} + \frac{b}{6} = 10 \quad \Rightarrow \quad b = 85$$

valor inicial $y(0^+)$ e o valor final $y(+\infty)$

$$Y(s) = \frac{6s^3 + 24s^2 + 12s + 60}{(s^2+4)(s+3)s}, \quad \mathrm{Re}(s) > 0$$

$$y(0^+) = \lim_{s \to +\infty} sY(s) = 6, \quad y(+\infty) = \lim_{s \to 0} sY(s) \nexists \qquad (\text{polos } \pm j2, 0, -3)$$

De fato, por frações parciais tem-se

$$y(t) = \big(5 + 3\cos(2t) - 2\exp(-3t)\big)\, u(t)$$

Note que

$$\lim_{s \to 0} sY(s) = 5 \qquad (\text{valor médio de regime})$$