# Resolução PR2 – 1º 2019

## Questão 1:

$$x[n] = ? \longrightarrow X(z) = \frac{z^2 - 13z}{(z+2)(z-3)} \qquad 2 < |z| < 3$$

→ Frações Parciais

$$\frac{X(z)}{z} = \frac{z-13}{(z+2)(z-3)} = \frac{\overset{=3}{A}}{(z+2)} + \frac{\overset{=-2}{B}}{(z-3)}$$

$$X(z) = \frac{3z}{(z+2)} - \frac{2z}{(z-3)}$$

→ Olhando para o domínio de existência temos

$2 < |z| < 3$, logo

$$X(z) = \underbrace{\frac{3z}{(z+2)}}_{\text{domínio positivo}} - \underbrace{\frac{2z}{(z-3)}}_{\text{domínio negativo}}$$

Transformadas

$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z-a} \quad |z| > |a|$$

$$\mathcal{Z}\{-a^n u[-1-n]\} = \frac{z}{z-a} \quad |z| < |a|$$

$$\boxed{x[n] = 2(3)^n u[-n-1] + 3(-2)^n u[n]}$$

## Questão 2: $\sum_{n=0}^{+\infty} \underbrace{(n2^{-n}+4^{-n})}_{x[k]} = ?$

Propriedade da SOMA:

$$\mathcal{Z}\{x[n]\}\Big|_{z=1} = \sum_{-\infty}^{+\infty} x[k]$$

Transformadas

$$\binom{n+m}{m} = \frac{(n+m)!}{m!\,(n)!}$$

- $\mathcal{Z}\{a^n u[n]\} = \dfrac{z}{z-a}$

- $\mathcal{Z}\left\{\binom{n}{m} a^{n-m} u[n]\right\} = \dfrac{z}{(z-a)^{m+1}}$

$$\hookrightarrow \mathcal{Z}\left\{\underbrace{\binom{n}{1}}_{n} a^{n-1} u[n]\right\} = \frac{z}{(z-a)^2}$$

$$\mathcal{Z}\{n2^{-n}+4^{-n}\} = \mathcal{Z}\{n(1/2)^n + (1/4)^n\}$$

$$\overset{!}{=} 1/2 \cdot \frac{z}{(z-1/2)^2} + \frac{z}{(z-1/4)}$$

$$\sum x[n] = \left(1/2 \cdot \frac{z}{(z-1/2)^2} + \frac{z}{(z-1/4)}\right)\Bigg|_{z=1} = \boxed{10/3}$$

## Questão 3:

$X$ e $Y$ são variáveis aleatórias discretas independentes

$$\mathcal{E}\{z^X\} = \sum_k z^k \Pr\{X=k\} = \frac{-4}{z-5} \quad |z|<5$$

$$\mathcal{E}\{z^Y\} = \sum_k z^k \Pr\{Y=k\} = \frac{-3}{2z-3} \quad |z|<5/2$$

a) $W = X+Y \longrightarrow \mathcal{E}\{z^W\} = ?$

Pela teoria temos que $\mathcal{E}\{z^W\} = \mathcal{E}\{z^X\}\,\mathcal{E}\{z^Y\}$

$$\boxed{\mathcal{E}\{z^X\}\mathcal{E}\{z^Y\} = \frac{12}{(z-5)(2z-5)} \quad |z|<5/2}$$

b) $\Pr\{W=0\} = ?$

$$\mathcal{E}\{z^X\}\,\mathcal{E}\{z^Y\} = \sum_k z^k \underbrace{\Pr\{X=k\}\Pr\{Y=k\}}_{\Pr\{W=k\}}$$

$$\sum_k z^k \Pr\{W=k\} = \mathcal{E}\{z^X\}\,\mathcal{E}\{z^Y\} \stackrel{!}{=} \frac{12}{(z-5)(2z-5)}$$

Se $k=0$

$$\Pr\{W=0\} = \left.\left(\frac{12}{(z-5)(2z-5)}\right)\right|_{z=0} \quad \boxed{\Pr\{W=0\} = \frac{12}{25}}$$

c) $\mathcal{E}\{W\} = ?$

$\rightarrow$ Pela teoria temos

- $\mathcal{E}\{X^m\} = \left(\frac{z\,d}{dz}\right)^m Z\{p[n]\}\Big|_{z=1}$

- $Z\{p[n]\} = \mathcal{E}\{z^X\}$

No nosso caso temos

$$\mathcal{E}\{W\} = z \cdot \frac{d}{dz}\left(Z\{p[n]\}\right)\Big|_{z=1}$$

$$Z\{p[n]\} = \mathcal{E}\{z^W\} = \underbrace{\mathcal{E}\{z^X\}\,\mathcal{E}\{z^Y\}}_{\text{item (a)}}$$

$$Z\{p[n]\} = \frac{12}{(z-5)(2z-5)}$$

$$\mathcal{E}\{W\} = z \cdot \frac{d}{dz}\left(\frac{12}{2z^2-15z+25}\right)\Big|_{z=1}$$

$$\overset{!}{=} z \cdot \left(\frac{-12\,(4z-15)}{(2z^2-15z+25)^2}\right)\Big|_{z=1}$$

$$\overset{!}{=} \frac{-12 \cdot (4-15)}{(2-15+25)^2} = \frac{-\cancel{12} \cdot (-11)}{\cancel{12}^2} = \boxed{\frac{11}{12}}$$

## Questão 4:

a) $Y(z) = ?$

$$y[n+2] + y[n+1] - 2y[n] = 0 \qquad y[0] \text{ e } y[1] \text{ dados}$$

Transformadas

- $\mathcal{Z}\{y[n+2]\} = z^2Y(z) - z^2y[0] - zy[1]$
- $\mathcal{Z}\{y[n+1]\} = zY(z) - zy[0]$
- $\mathcal{Z}\{y[n]\} = Y(z)$

$$z^2Y(z) - z^2y[0] - zy[1] + zY(z) - zy[0] - 2Y(z) = 0$$

$$Y(z)(z^2+z-2) = z^2y[0] + zy[1] + zy[0]$$

$$\boxed{Y(z) = \frac{z^2y[0] + z(y[1]+y[0])}{z^2+z-2}}$$

b) $y[n] = ?$ para $y[0] = 1$ $y[1] = 4$

$$Y(z) = \frac{z^2 \overbrace{y[0]}^{1} + z(\overbrace{y[0]}^{1} + \overbrace{y[1]}^{4})}{(z-1)(z+2)}$$

$$\stackrel{!}{=} \frac{z^2 + 5z}{(z-1)(z+2)}$$

Frações Parciais:

$$\frac{Y(z)}{z} = \frac{z+5}{(z-1)(z+2)} = \frac{A}{(z-1)} + \frac{B}{(z+2)} \qquad A=2 \quad B=-1$$

$$Y(z) = \frac{2z}{(z-1)} - \frac{z}{(z+2)}$$

Transformadas $\mathcal{Z}\{a^n u[n]\} = \frac{z}{z-a}$

$$\boxed{y[n] = (2(1)^n - (-2)^n)\,u[n]}$$

## Questão 5:

a) $y_f[n] = ?$

$$(p-2)(p-4)\,y[n] = 2^{n+3} \qquad (1)$$

$\alpha = 2 \quad \alpha = 4 \quad \gamma = 2$

raiz igual

$$y_f[n] = An(2)^n \qquad (2)$$

$(2) \rightarrow (1)$

$$(p^2 - 6p + 8)\,An(2)^n = 2^n \cdot 8$$

$$A(n+2)(2)^{n+2} - 6A(n+1)\cdot(2)^{n+1} + 8An(2)^n = 2^n \cdot 8$$

$$4A(n+2)2^n - 12A(n+1)2^n + 8An\,2^n = 2^n \cdot 8$$

$$2^n\left[n(\cancel{4A - 12A + 8A}) + (8A - 12A)\right] = 2^n \cdot 8$$

$\cancel{4A - 12A + 8A} \rightarrow 0$

$$-4A = 8$$

$$\boxed{A = -2}$$

$$\boxed{y_f[n] = -2n(2)^n}$$

b) $y[n] = ?$ p/ $y[0] = 1$ $y[1] = 10$

$$y[n] = y_f[n] + y_h[n]$$

já temos no item a

$$\hookrightarrow y_h[n] = B(2)^n + C(4)^n$$

$$y[n] = B.2^n + C\,4^n - 2n(2)^n$$

$$\hookrightarrow y[0] = 1 = B + C$$

$$y[1] = 10 = 2B + 4C - 4$$

$$\rightarrow \begin{cases} B + C = 1 & (\times 2) \\ 2B + 4C = 14 & + \end{cases}$$

$$2C = 12$$

$$\boxed{B = -5} \leftarrow \boxed{C = 6}$$

$$\boxed{y[n] = -5(2)^n + 6(4)^n - 2n(2)^n}$$

Questão 6:

$$\begin{cases} \dot{v}_1 = -2(v_1+2)v_2 - 5x \\ \dot{v}_2 = 5(v_2+3)v_1 + x \end{cases}$$

a) Pontos de equilíbrio $(\bar{v}_1, \bar{v}_2)$ para $x = 0$

→ Os pontos de equilíbrio são encontrados fazendo-se

$$\begin{cases} \dot{v}_1 = 0 \\ \dot{v}_2 = 0 \end{cases} \quad e \quad x = 0$$

Assim,

$$\begin{cases} -2(v_1+2)v_2 = 0 \rightarrow \text{Raízes} \begin{cases} v_1 = -2 \\ v_2 = 0 \end{cases} \\ 5(v_2+3)v_1 = 0 \rightarrow \text{Raízes} \begin{cases} v_1 = 0 \\ v_2 = -3 \end{cases} \end{cases}$$

Se $v_1 = 0 \rightarrow v_2 = 0 \rightarrow (0,0)$

Se $v_2 = -3 \rightarrow v_1 = -2 \rightarrow (-2,-3)$

$$\boxed{\begin{matrix} \text{Pontos de equilíbrio} \\ (0,0) \text{ e } (-2,-3) \end{matrix}}$$

b) Matrizes (A e b) do sistema linearizado

$$\dot{v} = Av + bx$$

$$A = \begin{bmatrix} \frac{\partial \dot{v}_1}{\partial v_1} & \frac{\partial \dot{v}_1}{\partial v_2} \\ \frac{\partial \dot{v}_2}{\partial v_1} & \frac{\partial \dot{v}_2}{\partial v_2} \end{bmatrix} \qquad B = \begin{bmatrix} -5 \\ 1 \end{bmatrix} x$$

$$\frac{\partial \dot{v}_1}{\partial v_1} = -2v_2 \qquad \frac{\partial \dot{v}_2}{\partial v_1} = 5v_2 + 15$$

$$\frac{\partial \dot{v}_1}{\partial v_2} = -2v_2 - 4 \qquad \frac{\partial \dot{v}_2}{\partial v_2} = 5v_1$$

$$A = \begin{bmatrix} -2v_2 & -2v_1 - 4 \\ -5v_2 + 15 & 5v_1 \end{bmatrix}$$

Para (0,0)

$$A\Big|_{(0,0)} = \begin{bmatrix} 0 & -4 \\ 15 & 0 \end{bmatrix}$$

Para (-2,-3)

$$A\Big|_{(\underbrace{-2}_{v_1},\underbrace{-3}_{v_2})} = \begin{bmatrix} 6 & 0 \\ 0 & -10 \end{bmatrix}$$

## Questão 7: Realização $(A, b, c, d) = ?$

$$\dddot{y} - 4\ddot{y} + 6\dot{y} - 5y = 2\dddot{x} - 10\ddot{x} + 15\dot{x} - 9x$$

↓ podemos escrever da forma equivalente como

$$y\,(p^3 - 4p^2 + 6p - 5) = (2p^3 - 10p^2 + 15p - 9)\,x$$

$$\frac{y}{x} = \frac{2p^3 - 10p^2 + 15p - 9}{p^3 - 4p^2 + 6p - 5}$$

grau do numerador = denominador

↳ CASO PRÓPRIO

Assim temos que encontrar uma expressão na forma

$$\frac{y}{x} = \frac{N(x)}{D(x)} + b \qquad \text{onde grau } N(x) < D(x)$$

$$\therefore \frac{2p^3 - 10p^2 + 15p + 9}{p^3 - 4p^2 + 6p - 5} = \underbrace{2}_{d} + \frac{\overbrace{-2p^2}^{\bar{\beta}_2} + \overbrace{3p}^{\bar{\beta}_1} + \overbrace{1}^{\bar{\beta}_0}}{p^3 \underbrace{-4p^2}_{\alpha_2} + \underbrace{6p}_{\alpha_1} \underbrace{-5}_{\alpha_0}}$$

FORMATO

$$\dot{v} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \cdots & & -\alpha_{m-1} \\ -\alpha_0 & & & & \end{bmatrix} v + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} \bar{\beta}_0 & \cdots & \bar{\beta}_{m-1} \end{bmatrix} v + \begin{bmatrix} d \end{bmatrix} x$$

Logo, nossa realização é:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -6 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} \quad d = \begin{bmatrix} 2 \end{bmatrix}$$

**Questão 8:** $y(t) = ?$

$$\dot{v} = \underbrace{\begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix}}_{A} v \qquad v(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$y = \underbrace{[3 \quad 1]}_{C} v$$

$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad d = 0$$

genericamente temos

$$\dot{v} = Av + bx \xrightarrow{\mathcal{L}} sV(s) - v[0] = AV(s) + bX(s) \quad [1]$$
$$y = Cv + dx \xrightarrow{\mathcal{L}} Y(s) = CV(s) + dX(s) \quad [2]$$

Manipulando [1] temos:

$$(s-A)V(s) - v[0] = bX(s)$$
$$(s-A)V(s) = bX(s) + v[0]$$
$$V(s) = (s-A)^{-1}\left(bX(s) + v[0]\right) \quad [3]$$

$[3] \rightarrow [2]$

$$\boxed{Y(s) = C(s-A)^{-1}(bX(s) + v[0]) + dX(s)}$$

Tendo a expressão acima podemos escrever

$Y(s)$

$$Y(s) = [3 \quad 1] \begin{bmatrix} s+4 & 4 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

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Obs:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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$$Y(s) = [3 \quad 1] \frac{1}{s^2+4s+4} \underbrace{\begin{bmatrix} s & -4 \\ 1 & s+4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}}_{\begin{bmatrix} 2s+8 \\ 2-2s-8 \end{bmatrix}}$$

$$Y(s) = \frac{1}{s^2+4s+4} \cdot 6s + 24 - 6 - 2s$$

$$= \frac{4s+18}{s^2+4s+4} = \frac{4s+18}{(s+2)^2}$$

$$= \frac{A^{=4}}{(s+2)} + \frac{B^{=10}}{(s+2)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{y(t) = (4e^{-2t} + 10te^{-2t})\,u(t)}$$

## Questão 9 :

$$\dot{\bar{v}} = \bar{A}\bar{v} \qquad \bar{v}(0) = \bar{v}_0 \qquad y = \bar{c}\bar{v}$$

$$y(t) = \underbrace{t\cos(3t) + 2t\sin(3t)}$$

$$\lambda = \pm 3j = \alpha \pm \beta j$$

multiplicidade 2

$\bar{A}$ na forma de Jordan

$$= \begin{bmatrix} \alpha & -\beta & 1 & 0 \\ \beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) & t\cos(\beta t) & -t\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) & t\sin(\beta t) & t\cos(\beta t) \\ 0 & 0 & \cos(\beta t) & -\sin(\beta t) \\ 0 & 0 & \sin(\beta t) & \cos(\beta t) \end{bmatrix}$$

$$y = \bar{c}\bar{v} = e^{At} \cdot v(0)$$

$$\bar{A} = \begin{bmatrix} 0 & -3 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad v(0) = e^{-At} y$$

$$e^{At} = \begin{bmatrix} \cos(3t) & -\sin(3t) & t\cos(3t) & -t\sin(3t) \\ \sin(3t) & \cos(3t) & t\sin(3t) & t\cos(3t) \\ 0 & 0 & \cos(3t) & -\sin(3t) \\ 0 & 0 & \sin(3t) & \cos(3t) \end{bmatrix}$$

$$t\cos(3t) + 2t\sin(3t) = e^{At} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}}_{v(0)}$$

refere-se à primeira linha $\bar{v}_1$

$$y = \bar{C} \cdot \bar{v} = e^{At} v(0)$$

$$\bar{C} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \bar{v}_1$$

logo,

$$\bar{A} = \begin{bmatrix} 0 & -3 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad \bar{C} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \bar{v}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

## Questão 10:

$$A = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ -4 & 4 & 4 \end{bmatrix} \qquad \Delta(\lambda) = \lambda^3 \to \lambda = 0$$

$$M = (A - \lambda I) = A \to \text{rank}(M) = 1$$

r = nº colunas − rank
$r = 3 - 1 = 2$ → numero de blocos de Jordan

$$M^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow 2 \to \text{dimensão do maior bloco}$$

$$\boxed{\text{diag}(J_2(0), J_1(0)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$J_1(\lambda) = [\lambda]$$

$$\to \text{diag}(J_2(\lambda), J_1(\lambda)) = \begin{bmatrix} J_2(\lambda) & 0 \\ 0 & J_1(\lambda) \end{bmatrix}$$