UNIVERSIDADE FEDERAL DE SANTA CATARINA

On Robustly Invariant Polyhedral Sets and Bilinear Programming for Designing Constrained Controllers

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Summary

1. Introduction
2. Set-theoretic output feedback control: A bilinear programming approach
3. Output feedback design for LPV systems subject to disturbances and control rate constraints
4. Conclusion

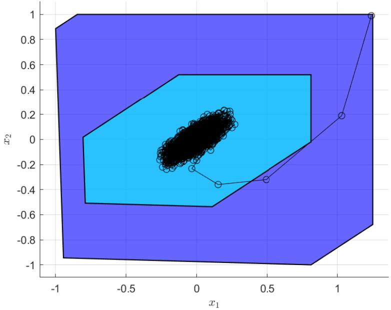
Summary

Introduction

- Considering physical and safety limits that occur in control systems is the primary concern of the *Constrained Control research* field
- *Set- invariance* and *contractivity* are fundamental concepts used to guarantee constraints fulfillment and determine regions of local stability [Tarbouriech et al., 2011, Blanchini and Miani, 2015]
- In practice, state and control constraints and exogenous disturbances are mostly bounded in amplitude and can be represented by *polyhedral sets*. Furthermore, *Output Feedback* (OF) control laws are often required in real-world applications
- **Objective**: **To show that Bilinear Programming is an effective optimization tool to design Output Feedback controllers for constrained LTI and LPV systems using Polyhedral Set-Invariance**

- *Set-invariance* properties relate convex (C-)sets (ellipsoidal, polyhedral, or composite) to a dynamical system (linear, nonlinear, LPV, or Fuzzy T-S) [Tarbouriech et al., 2011, Blanchini and Miani, 2015]

- For systems subject to persistent disturbances, the property of *Robust Positive Invariance* (RPI) ensures that any trajectory originating from a set within the state space will stay within that set. Additionally, if the set is *contractive*, the trajectory will ultimately be bounded within a subset surrounding the origin



- RPI reduces to the *Positive Invariance* property in the absence of disturbances, and the set contractivity guarantees the convergence of the system's trajectories to the origin [*Many authors, 20th Century*]

- *Robust Controlled Invariance* (RCI) ensures the existence of a control law that will make a set Robustly Positively Invariant

Lemma (Extended Farkas'Lemma (EFL) [Hennet, 1995])

Consider two polyhedral sets, $\mathcal{P}_i = \{x : P_i x \leq \phi_i\}$, $i = 1, 2$, $P_i \in \Re^{l_{p_i} \times n}$, *and positive vectors* $\phi_i \in \Re^{l_{p_i}}$. *Then* $\mathcal{P}_1 \subseteq \mathcal{P}_2$ *iff*

$$\begin{array}{r} P_2 x \leq \phi_2 \\ \forall x : P_1 x \leq \phi_1 \end{array} \iff \exists Q \geq 0 \ ; \ \begin{array}{rcl} Q\,P_1 & = & P_2 \\ Q\phi_1 & \leq & \phi_2 \end{array}$$

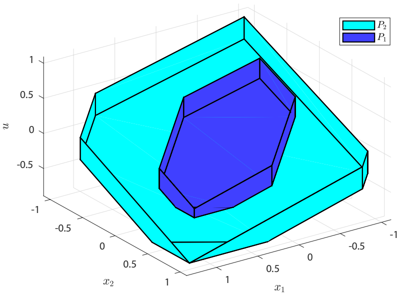


Figure 1: Set Inclusion

Definition

For a given the discrete-time system $x_{k+1} = (A + BKC)x_k$, the polyhedral set

$$\mathcal{L} = \{x_k \in \Re^n : Lx_k \leq \mathbf{1}_{l_r}\} \ , \ L \in \Re^{l_r \times n} \ , \ \mathbf{1}_{l_r} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T$$

is Positively Invariant and λ-contractive, with $\lambda \in [0\,,\,1)$, if and only if

$$\begin{matrix} Lx_{k+1} = L(A + BKC)x_k \leq \lambda \mathbf{1}_{l_g} \\ \forall\, Lx_k \leq \mathbf{1}_{l_g} \end{matrix} \iff \exists\, H \geq 0 \ ; \ \begin{matrix} HL & = & L(A + BKC) \\ H\mathbf{1}_{l_g} & \leq & \lambda \mathbf{1}_{l_g} \end{matrix}$$

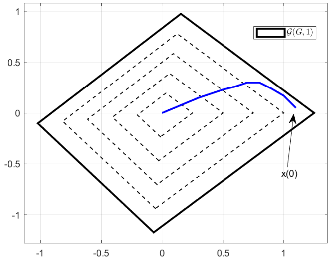


Figure 2: Positive Invariance with λ-contractivity

Glimpse on Positive Invariance and Bilinear Programming

- Brião et all. ***Explicit Computation of Stabilizing Feedback Control Gains Using Polyhedral Lyapunov Functions***. 2018 IEEE ICA-ACCA, Chile:

$$
\begin{aligned}
\underset{H,K,L,J,\lambda,\gamma}{\text{Min}} \quad & \lambda + \alpha\,\gamma \\
\text{s.t.} \quad & HL = L(A + BKC) \\
& \|H\|_\infty \leq \lambda \;,\; 0 \leq \lambda < 1 \\
& \|K\|_\infty \leq \gamma \;,\; 0 \leq \gamma \leq \overline{\gamma} \\
& LU = I_n \\
& (\underline{L}, \underline{U}) \leq (L, U) \leq (\overline{L}, \overline{U})
\end{aligned}
$$

Lyapunov function

$$v(x_k) = \|Lx_k\|_\infty \;,\; L \in \Re^{n \times n}$$

with $rank(L) = n$

- Lower and upper bounds on the unconstrained variables reduce the optimization search space $\Longrightarrow$ *Bilinear Program can be solved using nonlinear solvers, as* e.g. *KNITRO, which implements a multistart strategy to find local minima under convergence [Brião et al.(2021)]*

Related results for LTI and LPV systems

1. Brião; Castelan; Ernesto; Camponogara. ***Output feedback design for discrete-time constrained systems subject to persistent disturbances via bilinear programming***. Journal of the Franklin Institute, 2021.

   Asymmetrical constraints and disturbance bounds, Static and Dynamic OF design

2. Lucia; Ernesto; Castelan. ***Set-theoretic output feedback control: A bilinear programming approach.*** Automatica, 2023.

3. Ernesto, Castelan, Lucia. ***Control-rate Constrained Output Feedback Design for LPV Systems subject to Bounded Disturbance***. CBA 2024, Brazil.

Summary

Set-theoretic output feedback control: A bilinear programming approach

Lucia, Walter; Ernesto, Jackson G. and Castelan, Eugênio B.
In: **Automatica** 2023

Problem Formulation

Consider the LTI discrete-time system:

$$x_{k+1} = Ax_k + Bu_k + B_p p_k \quad (1a)$$

$$y_k = Cx_k + D_\eta \eta_k, \quad (1b)$$

$$k \in \mathbb{N} \,,\; x_k \in \Re^n,\; u_k \in \Re^m \,,\; y_k \in \Re^p,\; p_k \in \Re^s,\; \eta_k \in \Re^q$$

State and control constraints:

$$\mathcal{X} = \{x_k : Xx_k \leq \mathbf{1}_{l_x}\}, \text{ with } X \in \Re^{l_x \times n}, \quad (2a)$$

$$\mathcal{U} = \{u_k : Uu_k \leq \mathbf{1}_{l_u}\}, \text{ with } U \in \Re^{l_u \times m}, \quad (2b)$$

Bounded persistent disturbances:

$$\mathcal{P} = \{p_k : Pp_k \leq \mathbf{1}_{l_p}\}, \text{ with } P \in \Re^{l_p \times s}, \quad (3a)$$

$$\mathcal{N} = \{\eta_k : N\eta_k \leq \mathbf{1}_{l_n}\}, \text{ with } N \in \Re^{l_n \times r}. \quad (3b)$$

Definition (Robust Control Invariant (RCI) [Borrelli et al., 2017])

A set $\mathcal{Q} \subseteq \mathcal{X}$ is said RCI for the LTI discrete system under state and control constraints, also subject to the bounded persistent disturbances, if:

$$\begin{gathered} \forall x_k \in \mathcal{Q} \rightarrow \exists u_k \in \mathcal{U} : \\ Ax_k + Bu_k + B_p p_k \in \mathcal{Q}, \ \ \forall p_k \in \mathcal{P} \end{gathered} \tag{4}$$

Definition (Robustly One-Step Controllable (ROSC)[Borrelli et al., 2017])

Consider the LTI discrete system under state and control constraints, also subject to the bounded persistent disturbances, and a set $\mathcal{L}_i \subset \mathcal{X}$. The set of states ROSC to $\mathcal{L}_i$ in one-step, namely $\mathcal{L}_{i+1} \subseteq \mathcal{X}$, is defined as:

$$\mathcal{L}_{i+1} := \{ x \in \mathcal{X}, \exists u \in \mathcal{U} : Ax + Bu + B_p p \in \mathcal{L}_i, \ \forall p \in \mathcal{P} \} \tag{5}$$

Output feedback function:

$$u_k = f(y_k) \tag{6}$$

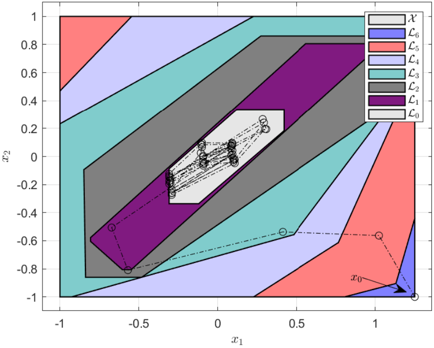
Problem

Find a stabilizing output feedback control function and an associated domain of attraction $\mathcal{L}_D \subseteq \mathcal{X}$, $0_n \in \mathcal{L}_D$ *such that* $\forall x_0 \in \mathcal{L}_D$ *and under the effect of the bounded persistent disturbances, the following properties are met:*

- $\mathcal{L}_D$ *is a RCI set.*
- *There exist a small RCI region* $\mathcal{L}_0 \subseteq \mathcal{L}_D$, $\mathbf{0}_n \in \mathcal{L}_0$ *where the state trajectory is ultimately bounded in a finite and a-priori known numbers of steps.*
- *The state and input constraints are fulfilled.*

— *Offline computations* —

1: Build a small terminal RCI region $\mathcal{L}_0$ and associated SoF gain K_0;
2: Build a family of $\bar{N}$ ROSC sets $\{\mathcal{L}_i\}$ and associated SoF controller gains $\{K_i\}$, until the set growth saturates;
3: Store $\{K_i, \mathcal{L}_i\}_{i=0}^{\bar{N}}$ for online use.



RCI and ROSC sets construction using $\mathcal{X}$

Defining

$$\mathcal{R}_i = \{x_k \in \Re : R_i x_k \leq \mathbf{1}_{r_i}\}, R_i \in \Re^{r_i \times n} \tag{7}$$

$\mathcal{L}_i$ is described as

$$\mathcal{L}_i = \{x_k \in \Re : L_i x_k \leq \mathbf{1}_{l_{r,i}}\}, L_i \in \Re^{l_{r,i} \times n}, rank(L_i) = n, \tag{8}$$

where, by construction,

$$L_i = \begin{bmatrix} R_i \\ \delta_i X \end{bmatrix} \text{ and } \mathbf{1}_{l_{r,i}} = \begin{bmatrix} \mathbf{1}_{r_i} \\ \mathbf{1}_{l_x} \end{bmatrix}, \tag{9}$$

with *set complexity* $l_{r,i} = r_i + l_x > n$ and $0 < \delta_i \leq 1, \forall\, i \implies \mathcal{L}_i \subseteq \mathcal{X}$.

RCI Bilinear Optimization Problem

$$\min_{\Gamma_0(.)\,,\,\delta_0} \quad \delta_0$$

s.t.

$$H_0 L_0 = L_0(A + BK_0C)$$
$$V_0 P = L_0 B_p$$
$$W_0 N = L_0 B K_0 D_\eta$$
$$H_0 \mathbf{1}_{l_{r,0}} + V_0 \mathbf{1}_{l_p} + W_0 \mathbf{1}_{l_\eta} \leq \mathbf{1}_{l_{r,0}}$$

$\Leftrightarrow$ $\mathcal{L}_0$ is RPI for the system controlled with $u_k = K_0 y_k$:

$$(A+BK_0C)\mathcal{L}_0 \oplus B_p\mathcal{P} \oplus BK_0D_\eta\mathcal{N} \subseteq \mathcal{L}_0$$

From EFL, $[\, H_0 \;\; V_0 \;\; W_0 \,] \geq 0$

$$M_0 L_0 = U K_0 C$$
$$Z_0 N = U K_0 D_\eta$$
$$M_0 \mathbf{1}_{l_{r,0}} + Z_0 \mathbf{1}_{l\eta} \leq \mathbf{1}_{lu}$$

$\Leftrightarrow$ Control constraints admissibilty

$$K_0 C\mathcal{L}_0 \oplus K_0 D_\eta \mathcal{N} \subseteq \mathcal{U}$$

$$[\, Z_0 \;\; M_0 \,] \geq 0$$

$$T_0 S = L_0 \quad , \quad T_0 \mathbf{1}_{l_s} \leq \mathbf{1}_{l_{r,0}}$$

$\Leftrightarrow$ $\mathcal{S} \subseteq \mathcal{L}_0$ for *good conditioning*

$$J_0 L_0 = I_n \quad , \quad 0 < \delta_0 \leq 1$$

$\Leftrightarrow$ $rank(L) = n$ and $\mathcal{L}_0 \subseteq \delta_0 \mathcal{X}$

$$\underline{\Gamma}_0(.) \leq \Gamma_0(.) \leq \bar{\Gamma}_0(.)$$

$\Rightarrow$ To bound unconstrained variables

ROSC Bilinear Optimization Problem - $i = 1, \ldots, \bar{N}$

$$\max_{\Gamma_i(.),\,\delta_i,\,\gamma_t} \quad \mathcal{J}_i = \sum_{t=1}^{n} \gamma_t$$

s.t.

$$H_i L_i = L_{i-1}(A + BK_i C)$$
$$V_i P = L_{i-1} B_p$$
$$W_i N = L_{i-1} B K_i D_\eta$$
$$H_i \mathbf{1}_{l_r} + V_i \mathbf{1}_{l_p} + W_i \mathbf{1}_{l_\eta} \le \mathbf{1}_{l_r}$$

$\Leftrightarrow$ $\mathcal{L}_i \subseteq \mathcal{X}$ is ROSC to $\mathcal{L}_{i-1}$ for the system controlled with $u_k = K_i y_k$:

$$(A+BK_iC)\mathcal{L}_i \oplus B_p\mathcal{P} \oplus BK_iD_\eta\mathcal{N} \subseteq \mathcal{L}_{i-1}$$

From EFL, $[\, H_i \;\; V_i \;\; W_i \,] \ge 0$

$$M_i L_i = U K_i C \;,\;\; Z_i N = U K_i D_\eta$$
$$M_i \mathbf{1}_{l_r} + Z_i \mathbf{1}_{l_\eta} \le \mathbf{1}_{l_u}$$

$\Leftrightarrow$ Control constraints admissibility:

$$K_iC\mathcal{L}_i \oplus K_iD_\eta\mathcal{N}\mathcal{L}_i \subseteq \mathcal{U}$$
$$[\, Z_i \;\; M_i \,] \ge 0$$

$$T_i L_{i-1} = L_i \;,\;\; \delta_{i-1} < \delta_i \le 1$$
$$T_i \mathbf{1}_{l_r} \le \mathbf{1}_{l_r} \;,\;\; T_i \ge 0$$

$\Leftrightarrow$ Recursive sets inclusion: $\mathcal{L}_{i-1} \subseteq \mathcal{L}_i$

$$\gamma_t L_i v_t \;,\; t = 1, \ldots, t$$

$\Rightarrow$ To enlarge $\mathcal{L}_i$ in given directions v_t

$$J_i L_i = I_n \;,\;\; \underline{\Gamma_i}(.) \le \Gamma_i(.) \le \overline{\Gamma_i}(.)$$

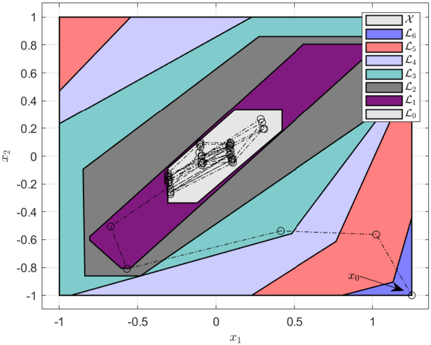
$\Leftrightarrow$ $rank(L) = n$ and bounded variables

— *Online switching rule* —
($\forall\, k,\, x_0 \in \mathcal{L}_D$)

1: Given y_k, compute i_k using Propositions 1 and 2 in [Lucia et al., 2023]:

$$i_k = \begin{cases} \bar{i}_k & \textbf{if } rank(C) < n \\ \underline{i}_k & \textbf{if } rank(C) = n \end{cases}$$

2: Compute and apply $u_k = K_{i_k} y_k$.



Offline numerical complexity

Number of variables, equality and inequality constraints in the RCI and ROSC optimization problems

	RCI set $\mathcal{L}_0$
# of Variables	$mp + l_0(n^2 + l_0 + l_p + l_n + l_u + l_s) + l_u l_n$
# of Equalities	$l_0(n^2 + s + r) + l_u(n + r) + n^2$
# of Inequalities	$l_0 + l_u + l_s$

	ROSC sets $\mathcal{L}_i$
# of Variables	$mp + l_{i-1}(n + l_i^2 + l_p + l_n) + l_u(l_i + l_n) + nl_i$
# of Equalities	$l_{i-1}(n + s + r) + l_u(n + r) + l_i n + n^2$
# of Inequalities	$l_{i-1} + l_u + l_i$

Example

Consider the Double Integrator system:

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} p_k \tag{10a}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \eta_k, \tag{10b}$$

subject to

$$-1 \leq x_{k,1} \leq 1.25 \ , \ |x_{k,2}| \leq 1 \ , \ -0.8 \leq u_k \leq 1 \ , \ |p_k| \leq 0.1 \ , \ |\eta_k| \leq 0.1$$

which implies the matrices

$$X^T = \begin{bmatrix} 0.8 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \ , \quad U^T = \begin{bmatrix} 1 & -1.25 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 10 & -10 \end{bmatrix}^T \ , \quad N^T = \begin{bmatrix} 10 & -10 \end{bmatrix}$$

i	K_i	$\mathcal{L}_i$ Area	$\mathcal{J}_i$
0	[-0.7500]	0.3120	2.3466
1	[-0.7803]	0.4054	2.7227
2	[-0.8686]	0.7949	4.1941
3	[-0.8476]	1.7372	5.9305
4	[-0.6979]	2.5962	7.3990
5	[-0.6111]	3.0652	8.0587
6	[-0.6111]	3.1510	8.2741

Table 1: ST-OF, offline design.

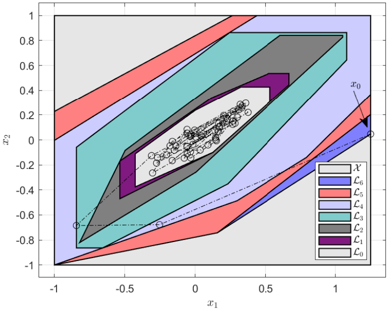


Figure 3: ST-OF: DoA for ($\bar{t} = 8,\ r = 4,$), and state trajectory for $x_0 = [1.25,\ 0.047]^T \in \mathcal{L}_6$.

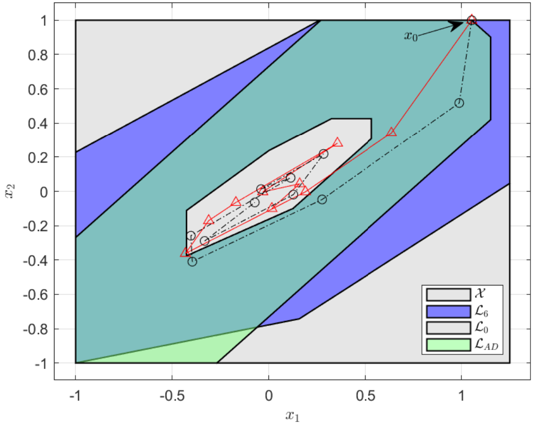


Figure 4: DoA and state trajectory: ST-OF ○ vs △ De Almeida and Dorea [2020] (online optimization to find u_k)

$$Area_{\mathcal{L}_6} = 3.1510 \quad vs \quad Area_{AD} = 2.4837$$

Summary

Control-rate Constrained Output Feedback Design for LPV Systems Subject to Disturbances

Ernesto, Jackson G., Eugênio B. Castelan e Walter Lucia.
In: **CBA 2024**.

Problem Formulation

LPV discrete-time system

$$x_{k+1} = A(\alpha_k)x_k + B(\alpha_k)u_k + B_p(\alpha_k)p_k \quad (11a)$$

$$y_k = Cx_k + D_\eta \eta_k \quad (11b)$$

$$\begin{bmatrix} A(\alpha_k) & B(\alpha_k) & B_p(\alpha_k) \end{bmatrix} = \sum_{i=1}^{\nu} \alpha_{k,i} \begin{bmatrix} A_i & B_i & B_{pi} \end{bmatrix}, \quad \alpha_k \in \mathcal{S}_{implex}$$

State, control input and rate variation constraints:

$$\mathcal{X} = \{x_k : Xx_k \leq \mathbf{1}_{l_x}\}, \quad X \in \Re^{l_x \times n_x} \quad (12a)$$

$$\mathcal{U} = \{u_k : Uu_k \leq \mathbf{1}_{l_u}\}, \quad U \in \Re^{l_u \times n_u} \quad (12b)$$

$$\mathcal{U}_d = \{\delta u_k : U_d \delta u_k \leq \mathbf{1}_{l_d}\}, \quad U_d \in \Re^{l_d \times n_u}, \quad \delta u_k = u_{k+1} - u_k \quad (12c)$$

Bounded persistent disturbances:

$$\mathcal{P} = \{p_k : Pp_k \leq \mathbf{1}_{l_p}\}, \quad P \in \Re^{l_p \times n_p} \quad (13a)$$

$$\mathcal{N} = \{\eta_k : N\eta_k \leq \mathbf{1}_{l_n}\}, \quad N \in \Re^{l_n \times n_\eta} \quad (13b)$$

Augmented state vector:

$$\xi_k = \begin{bmatrix} x_k^T & u_k^T \end{bmatrix}^T \in \Re^{n_\xi}, \; n_\xi = n_x + n_u \tag{14}$$

Augmented output vector:

$$\upsilon_k = \begin{bmatrix} y_k^T & u_k^T & y_{k+1}^T \end{bmatrix}^T \in \Re^{n_\upsilon}, \; n_\upsilon = 2n_y + n_u \tag{15}$$

Parameter-varying control increment input vector:

$$\delta u_k = \begin{bmatrix} K(\alpha_k) & \bar{K}(\alpha_k) & \hat{K} \end{bmatrix} \begin{bmatrix} y_k \\ u_k \\ y_{k+1} \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} K(\alpha_k) & \bar{K}(\alpha_k) & \hat{K} \end{bmatrix} = \sum_{i=1}^{\nu} \alpha_{k,i} \begin{bmatrix} K_i & \bar{K}_i & \hat{K} \end{bmatrix}$$

$$K_i \in \Re^{n_u \times n_y}, \; \bar{K}_i \in \Re^{n_u \times n_u}, \; \forall i = 1, \ldots, \nu, \text{ and } \hat{K} \in \Re^{n_u \times n_y}$$

Closed-loop augmented system:

$$\xi_{k+1} = \mathbb{A}^{cl}(\alpha_k)\xi_k + \mathbb{B}_d^{cl}(\alpha_k)d_k \tag{17}$$

$$\begin{bmatrix} \mathbb{A}^{cl}(\alpha_k) & \mathbb{B}_d^{cl}(\alpha_k) \end{bmatrix} = \sum_{i=1}^{\nu} \alpha_{k,i} \begin{bmatrix} \mathbb{A}_i^{cl} & \mathbb{B}_{d,i}^{cl} \end{bmatrix}$$

$$\mathbb{A}_i^{cl} = \begin{bmatrix} A_i & B_i \\ (K_i C + \hat{K} C A_i) & (\bar{K}_i + \hat{K} C B_i) + I \end{bmatrix} = \begin{bmatrix} E_i \\ \mathbb{E}_i + [0 \ \ I] \end{bmatrix}$$

$$\mathbb{B}_{d,i}^{cl} = \begin{bmatrix} B_{p,i} & 0 & 0 \\ \hat{K} C B_{p,i} & K_i D_\eta & \hat{K} D_\eta \end{bmatrix} = \begin{bmatrix} F_i \\ \mathbb{F}_i \end{bmatrix} \quad , \ d_k = \begin{bmatrix} p_k \\ \eta_k \\ \eta_{k+1} \end{bmatrix} \in \Re^{n_d}$$

Augmented state constraints:

$$\Xi = \{\xi_k : \mathbb{X}\xi_k \leq \mathbf{1}_{l_\xi}\}, \ \Xi = \begin{bmatrix} X & 0 \\ 0 & U \end{bmatrix} \in \Re^{l_\xi \times n_\xi}, \tag{18}$$

Augmented bounded disturbance:

$$\Delta = \{d_k : \mathbb{D}d_k \leq \mathbf{1}_{l_\Delta}\}, \ \mathbb{D} = \begin{bmatrix} P & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix} \in \Re^{l_\Delta \times n_d} \tag{19}$$

Definition

$\mathcal{L} \in \Re^{n_\xi}$ is a contractive robust positive invariant (RPI-)set of the LPV system, with ultimately bounded (UB-)set $\mathcal{L}_0 \subseteq \mathcal{L}$, if for any $\xi_0 = \begin{bmatrix} x_0^T & u_0^T \end{bmatrix}^T \in \mathcal{L}$ and $d_k = \begin{bmatrix} p_k^T & \eta_k^T & \eta_{k+1}^T \end{bmatrix}^T \in \Delta$, the corresponding state trajectory remains inside $\mathcal{L}$, converge to $\mathcal{L}^0$ in a finite number of steps, and remains ultimately bounded within $\mathcal{L}^0$, for all $\alpha_k \in \mathcal{S}$.

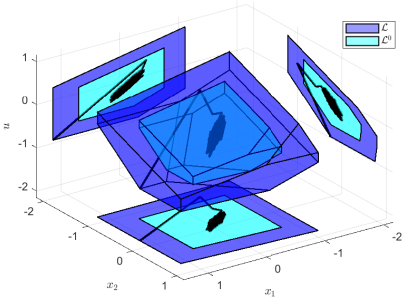
Consider the polyhedral sets:

$$\mathcal{L} = \{\xi_k : \mathbb{L}\xi_k \leq \mathbf{1}_{l_r}\},$$

$$\mathcal{L}^0 = \{\xi_k : \mathbb{L}\xi_k \leq \rho\mathbf{1}_{l_r}\}$$

$$\mathbb{L} \in \Re^{l_r \times n_\xi}, \; rank(\mathbb{L}) = n_\xi,$$

set complexity $l_r > n$, and $0 < \rho \leq 1$



Problem

Given l_r, find control gains $(K_i, \bar{K}_i \; \hat{K})$ and a triplet $(\mathbb{L}, \lambda, \rho)$, which defines a large *contractive RPI set $\mathcal{L} \subseteq \Xi$ and a* small *UB-set $\mathcal{L}^0 \subseteq \mathcal{L}$, such that, for any initial condition $\xi_0 \in \mathcal{L}$, $d_k \in \Delta$, and for all $\alpha_k \in \mathcal{S}$, the state, control, and control-rate variation constraints, $\mathcal{U}_d = \{U_d \delta u_k \leq \mathbf{1}_{l_d}\}$, are fulfilled.*

Maximizing the size of $\mathcal{L}$

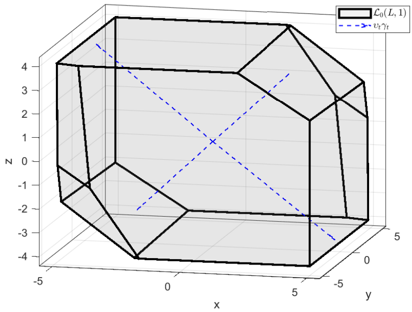
Auxiliary inequalities:

$$\gamma_t \mathbb{L} \psi_t \leq \mathbf{1}_{l_r},\ t = 1, \ldots, \bar{t} \tag{20}$$

where $\gamma_t \in \Re$ are positive scaling factors associated to the a pre-defined set of $\bar{t}$ directions

$$\Psi = \{\gamma_t \psi_t,\ t = 1, \ldots, \bar{t}\} \tag{21}$$

with $\psi_t = \begin{bmatrix} \psi_{x,t}^T & \psi_{u,t}^T \end{bmatrix}^T$, $\psi_{x,t} \in \Re^{n_x}$ and $\psi_{u,t} \in \Re^{n_u}$, which can be set as a variable $\psi_{u,t}$



Bilinear Optimization Problem

$$\max_{\Gamma(.)} \quad \sum_{t=1}^{\bar{t}} \gamma_t \; - \alpha\rho$$

s.t.

$H_i\mathbb{L} = \mathbb{L}\mathbb{A}_i^{cl} \;,\; H_i \geq 0$

$V_i\mathbb{D} = \mathbb{L}\mathbb{B}_{d,i}^{cl} \;,\; V_i \geq 0$ $\quad\Leftrightarrow$ RPI of $\mathcal{L}$, with λ-contractivity

$H_i\mathbf{1}_{l_r} + V_i\mathbf{1}_{l_\Delta} \leq \lambda\mathbf{1}_{l_r}$

$H_i\rho\mathbf{1}_{l_r} + V_i\mathbf{1}_{l_\Delta} \leq (1-\epsilon)\rho\mathbf{1}_{l_r}$ $\quad\Leftrightarrow$ RPI of the UB-set $\mathcal{L}^0 \subseteq \mathcal{L}$

$G\mathbb{L} = \mathbb{X} \;,\; G \geq 0$

$G\mathbf{1}_{l_r} \leq \mathbf{1}_{l_\xi}$ $\quad\Leftrightarrow$ $\mathcal{L} \subseteq \Xi$: state and control constraints fulfilment

$Q_i\mathbb{L} = U_d\mathbb{E}_i \;,\; Q_i \geq 0$

$T_i\mathbb{D} = U_d\mathbb{F}_i \;,\; T_i \geq 0$ $\quad\Leftrightarrow$ Control increment admissibility

$Q_i\mathbf{1}_{l_r} + T_i\mathbf{1}_{l_\Delta} \leq \mathbf{1}_{l_d}$

$\mathbb{J}\mathbb{L} = I_{n_\xi} \;,\; \gamma_t\mathbb{L}\psi_t \leq \mathbf{1}_{l_r}$ $\quad\Leftrightarrow$ $rank(\mathbb{L}) = n_\xi$ and set enlargement in given directions

$\underline{\Gamma}(.) \leq \Gamma(.) \leq \overline{\Gamma}(.)$

Example

LPV discrete-time system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} [\mathbf{2}\,,\,\mathbf{2.25}] \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} p_k \quad (22a)$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \eta_k, \quad (22b)$$

subject to

$$-1 \leq x_{k,1} \leq 1.25\ , \quad |x_{k,2}| \leq 1\ , \quad -0.8 \leq u_k \leq 1\ , \quad |p_k| \leq 0.1\ , \quad |\eta_k| \leq 0.1$$

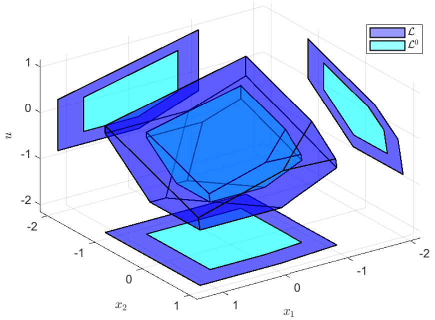
which implies the matrices

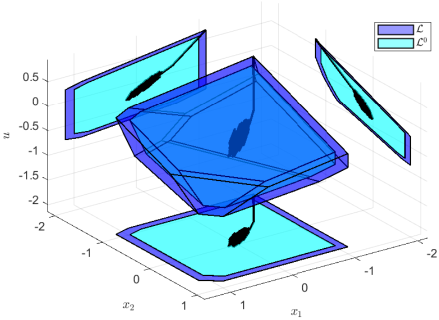
$$X^T = \begin{bmatrix} 0.8 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\ , \quad U^T = \begin{bmatrix} 1 & -1.25 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 10 & -10 \end{bmatrix}^T\ , \quad N^T = [\ 10 \quad -10\ ]$$

Table 2: Design with $l_r = 9$, $\bar{t} = 16$, $\alpha = 10$, and $\psi_{u,t}$ as a variable

δu_k bounds	$\mathcal{L}$ Vol.	Pr.Area	ρ	$[K_i \quad \bar{K}_i \quad \hat{K}]$
without	2.4217	4.4656	0.6710	$\begin{bmatrix}0.4431 & -0.5420 & -0.7376\end{bmatrix}$ $\begin{bmatrix}0.4661 & -0.4377 & -0.7376\end{bmatrix}$
$[-0.9\ ,\ 0.6]$	1.3584	4.4679	0.8864	$\begin{bmatrix}0.4033 & -0.5366 & -0.6016\end{bmatrix}$ $\begin{bmatrix}0.4033 & -0.4317 & -0.6016\end{bmatrix}$
$[-0.7\ ,\ 0.5]$	1.1997	4.4028	0.9980	$\begin{bmatrix}0.3743 & -0.5575 & -0.5551\end{bmatrix}$ $\begin{bmatrix}0.3734 & -0.4533 & -0.5551\end{bmatrix}$





Improved $\delta u_k = K(\alpha_k)y_k + \bar{K}(\alpha_k)u_k + \hat{K}(\alpha_{k+1})y_{k+1}$

- Improved $\delta u_k = u_{k+1} - uk$ implies

  $\mathbb{A}^{cl}(\alpha_k, \alpha_{k+1})$ and $\mathbb{B}^{cl}(\alpha_k, \alpha_{k+1})$

- Modified $\mathcal{L}_0 = \{\xi_k \; ; \; \mathbb{L}\xi_k \leq \boldsymbol{\rho}\}$, with

$$\boldsymbol{\rho} = [\;\rho_1 \quad \ldots \quad \rho_{l_r}\;]^T$$

- Modified objective function

$$\text{Max} \sum_{t=1}^{\bar{t}} \frac{\gamma_t}{\bar{t}} - \alpha \sum_{i=1}^{l_r} \frac{\boldsymbol{\rho}_i}{l_r}$$

- Numerical complexity increases but less conservative results are obtained

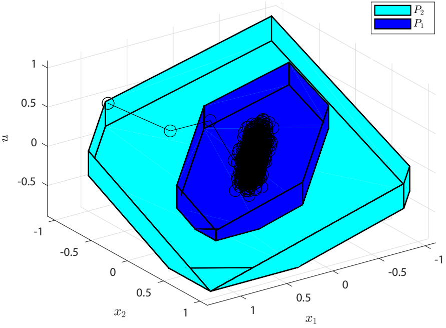


Figure 7

Summary

Conclusion

- Bilinear programming is an effective optimization tool to design output feedback controllers for constrained LTI and LPV systems through polyhedral set-invariance
- Bilinear optimization problems were solved using the KNITRO solver - Artelys. Free access from `https://neos-server.org/neos/`
- Explicit computation of the control feedback matrices allows for specific consideration of control gain structures and the design of reduced-order dynamical controllers and decentralized control laws
- Ongoing collaborations: time-delay and second-order systems, *PID-like* control design for reference tracking and disturbance rejection
- For dealing with the numerical complexity issue in higher-dimensional and *Complex Systems*, one can explore the system and controller structures

Further publications

1. Isidório; Dórea; Castelan. ***Observer-Based Output Feedback Control Using Invariant Polyhedral Sets for Fuzzy T–S Models Under Constraints***. Journal of Control, Automation, and Electrical Systems, 2023.

2. França; Castelan; Lucia. ***On the Design of Constrained PI-Like Output Feedback Tracking Controllers via Robust Positive Invariance and Bilinear Programming***. IEEE Control Systems Letter, 2023.

3. Vargas; Araújo; Dórea; Castelan. ***Eigenstructure assignment of constrained second-order linear systems: an optimised positively invariant set control approach***. Int. J. of Systems Science, 2024.

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